11.10 SOLUTION OF TWO DIMENSIONAL HEAT EQUATION

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad ...(1)$$

The methods employed for the solution of one dimensional heat equation can be readily extended to the solution of (1).

Consider a square region $0 \le x \le y \le a$ and assume that u is known at all points within and on the boundary of this square.

If h be the step-size then a mesh point $(x, y, t) = (ih, jh, nl)$ may be denoted as simply (i, j, n).

Replacing the derivatives in (1) by their finite difference approximations, we get

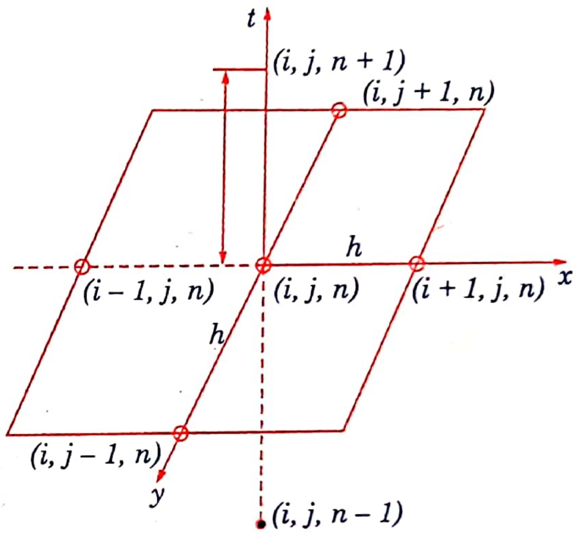

Fig. 11.29

$$\frac{u_{i,j,n+1} - u_{i,j,n}}{l}$$

$$= \frac{c^2}{h^2} \{(u_{i-1,j,n} - 2u_{i,j,n} + u_{i+1,j,n})$$

$$+ (u_{i,j-1,n} - 2u_{i,j,n} + u_{i,j+1,n})\}$$

i.e. $$u_{i,j,n+1} = u_{i,j,n} + \alpha(u_{i-1,j,n} + u_{i+1,j,n} + u_{i,j+1,n} + u_{i,j-1,n} - 4u_{i,j,n}) \quad ...(2)$$

where $\alpha = lc^2/h^2$. This equation needs the five points available on the nth plane (Fig. 11.29).

The computation process consists of point-by-point evaluation in the $(n + 1)$th plane using the points on the nth plane. It is followed by plane-by-plane evaluation. This method is known as ADE *(Alternating Direction Explicit) method.*

■ **Example 11.13.** *Solve the equation* $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ *subject to the initial conditions* $u(x, y, 0) = \sin 2\pi x \sin 2\pi y$, $0 \le x, y \le 1$, *and the conditions* $u(x, y, t) = 0$, $t > 0$ *on the boundaries, using ADE method with* $h = \frac{1}{3}$ *and* $\alpha = \frac{1}{8}$. *(Calculate the results for one time level).*

Sol. The equation (2) above becomes

$$u_{i,j,n+1} = u_{i,j,n} + \tfrac{1}{8}\,(u_{i-1,j,n} + u_{i+1,j,n} + u_{i,j+1,n} + u_{i,j-1,n} - 4u_{i,j,n})$$

i.e.
$$u_{i,j,n+1} = \tfrac{1}{2}\,u_{i,j,n} + \tfrac{1}{8}\,(u_{i-1,j,n} + u_{i+1,j,n} + u_{i,j+1,n} + u_{i,j-1,n}) \qquad ...(1)$$

The mesh points and the computational model is given in Fig. 11.30.

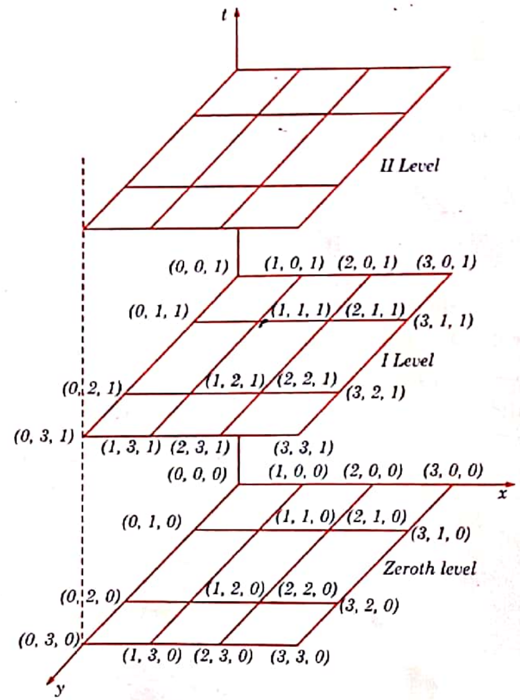

Fig. 11.30

At the zeroth level $(n = 0)$, the initial and boundary conditions are

$$u_{i,j,0} = \sin\frac{2\pi i}{3}\sin\frac{2\pi j}{3}$$

and
$$u_{i,0,0} = u_{0,j,0} = u_{3,j,0} = u_{i,3,0} = 0\ ;\ i, j = 0, 1, 2, 3.$$

Now we calculate the mesh values at the first level :

For $n = 0$, (1) gives

$$u_{i,j,1} = \tfrac{1}{2}\,u_{i,j,0} + \tfrac{1}{8}\,(u_{i-1,j,0} + u_{i+1,j,0} + u_{i,j+1,0} + u_{i,j-1,0}) \qquad ...(2)$$

(*i*) Put $i = j = 1$ in (2) :

$$u_{1,1,1} = \tfrac{1}{2}\,u_{1,1,0} + \tfrac{1}{8}\,(u_{0,1,0} + u_{2,1,0} + u_{1,2,0} + u_{1,0,0})$$
$$= \frac{1}{2}\left(\sin\frac{2\pi}{3}\right)^2 + \frac{1}{8}\left(0 + \sin\frac{4\pi}{3}\sin\frac{2\pi}{3} + \sin\frac{2\pi}{3}\sin\frac{4\pi}{3} + 0\right)$$
$$= \frac{3}{8} + \frac{1}{8}\left(-\frac{\sqrt{3}}{2}\times\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\times\frac{\sqrt{3}}{2}\right) = \frac{3}{16}$$

(*ii*) Put $i = 2, j = 1$ in (2)

$$u_{2,1,1} = \tfrac{1}{2}\,u_{2,1,0} + \tfrac{1}{8}\,(u_{1,1,0} + u_{3,1,0} + u_{2,2,0} + u_{2,0,0})$$
$$= \frac{1}{2}\sin\frac{4\pi}{3}\sin\frac{2\pi}{3} + \frac{1}{8}\left\{\left(\sin\frac{2\pi}{3}\right)^2 + 0 + \left(\sin\frac{4\pi}{3}\right)^2 + 0\right\}$$
$$= -\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{8}\left\{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2\right\} = -\frac{3}{16}.$$

(*iii*) Put $i = 1, j = 2$ in (2) :

$$u_{1,2,1} = \tfrac{1}{2}\,u_{1,2,0} + \tfrac{1}{8}\,(u_{0,2,0} + u_{2,2,0} + u_{1,1,0})$$
$$= \frac{1}{2}\sin\frac{2\pi}{3}\sin\frac{4\pi}{3} + \frac{1}{8}\left\{0 + \left(\sin\frac{4\pi}{3}\right)^2 + 0 + \left(\sin\frac{2\pi}{3}\right)^2\right\}$$
$$= -\frac{3}{8} + \frac{1}{8}\left(\frac{3}{4} + \frac{3}{4}\right) = -\frac{3}{16}$$

(*iv*) Put $i = 2, j = 2$ in (2) :

$$u_{2,2,1} = \tfrac{1}{2}\,u_{2,2,0} + \tfrac{1}{8}\,(u_{1,2,0} + u_{3,2,0} + u_{2,3,0} + u_{2,1,0})$$
$$= \frac{1}{2}\left(\sin\frac{4\pi}{3}\right)^2 + \frac{1}{8}\left(\sin\frac{2\pi}{3}\sin\frac{4\pi}{3} + 0 + 0 + \sin\frac{4\pi}{3}\sin\frac{2\pi}{3}\right)$$
$$= \frac{3}{8} + \frac{1}{8}\left(-\frac{3}{4} - \frac{3}{4}\right) = -\frac{3}{16}$$

Similarly the mesh values at the second and higher levels can be calculated.

PROBLEMS 11.4

1. Find the solution of the parabolic equation $u_{xx} = 2u_t$ when $u(0, t) = u(4, t) = 0$ and $u(x, 0) = x(4 - x)$, taking $h = 1$. Find the values upto $t = 5$.

2. Solve the equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ with the conditions $u(0, t) = 0$, $u(x, 0) = x(1 - x)$ and $u(1, t) = 0$. Assume $h = 0.1$. Tabulate u for $t = k$, $2k$ and $3k$ choosing an appropriate value of k. *(Anna, B.E., 2004)*

3. Given $\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial t} = 0$; $f(0, t) = f(5, t) = 0$, $f(x, 0) = x^2(25 - x^2)$; find the values of f for $x = ih$ $(i = 0, 1, ..., 5)$ and $t = jk$ $(j = 0, 1, ..., 6)$ with $h = 1$ and $k = \frac{1}{2}$, using the explicit method. *(Anna, B. Tech., 2012)*

4. Given $\partial u/\partial t = \partial^2 u/\partial t^2$, $u(0, t) = 0$, $u(4, t) = 0$ and $u(x, 0) = \frac{x}{3}(16 - x^2)$. Obtain $u_{i, j}$ for $10 = 1, 2, 3, 4$ and $j = 1, 2$ using Crank-Nicholson's method.

5. Solve the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ subject to the conditions $u(0, t) = u(1, t) = 0$ and

$$u(x, 0) = \begin{cases} 2x & \text{for} \quad 0 \le x \le 1/2 \\ 2(1 - x) & \text{for} \quad 1/2 \le x \le 1 \end{cases}$$

Take $h = 1/4$ and k according to Bandre-Schmidt equation.

6. Solve the 2-dimensional heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ satisfying the initial condition : $u(x, y, 0) = \sin \pi x \sin \pi y$, $0 \le x, y \le 1$ and the boundary conditions : $u = 0$ at $x = 0$ and $x = 1$ for $t > 0$. Obtain the solution upto two time levels with $h = \frac{1}{3}$ and $\alpha = \frac{1}{8}$.